

One-Loop Splitting Amplitudes in Gauge Theory

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Abstract

We recompute the functions describing the collinear factorization of one-loop amplitudes using the unitarity-based method. We present the results in a form suitable for use as an ingredient in two-loop calculations. We also present a function summarizing the behavior at one loop in both the soft and collinear limits.

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1. Introduction

The properties of non-Abelian gauge-theory amplitudes in soft and collinear limits play an important role both in formal proofs and in explicit calculations. The singularities that arise in these limits create technical obstacles to calculations of amplitudes and cross sections. Their universal structure, however, provides a means of resolving these problems. The universality of the limits is furthermore essential to the applicability of quantum chromodynamics to short-distance physics. It also provides a practical means for checking calculations of higher-point amplitudes. Explicit calculations of the functions governing these limits thus play an important role in the program of precision quantum chromodynamical calculations.

In these singular limits, on-shell amplitudes factorize into sums of lower-point amplitudes multiplied by universal functions. In the collinear limit, these functions are called *splitting amplitudes*. At tree level, one may derive the splitting amplitudes from a string representation [1] or from the Berends–Giele recurrence relations [2]. The squares of these tree-level functions, summed over helicities, yield the Altarelli–Parisi kernels [3,4].

Beyond leading order, the splitting amplitudes do not yield the Altarelli–Parisi kernel directly, and the relation between the two quantities remains to be clarified. The explicit forms of the splitting amplitudes at one-loop through $\mathcal{O}(\epsilon^0)$ (in dimensional regularization, with $D = 4 - 2\epsilon$) have previously been extracted by taking the collinear limit explicitly in various five-point amplitudes [5,6], or from an analysis of one-loop integrals [7]. In addition, Bern, Del Duca, and Schmidt [8] have recently given an expression for the gluon splitting amplitude to all orders in the dimensional regulator ϵ .

Many of the complicated one-loop computations of recent years have been performed using the unitarity-based method developed by Bern, Dixon, Dunbar, and one of the authors [9,10]. The method also leads to a concise proof [11] of collinear factorization to all loop orders in gauge theories, and as a by-product provides a compact formula for computing the splitting amplitudes explicitly. In this paper, we use this formula to compute the various one-loop splitting amplitudes, thus providing complete forms to all orders in ϵ for both gluon and fermion external states. The splitting amplitudes presented here will be useful in computations of physical observables such as jet cross-sections, at next-to-next-to-leading order. In addition to the one-loop splitting amplitudes, for such purposes one would also need the tree-level functions describing factorization when three particles become collinear, computed by Campbell and Glover [12] and by Catani and Grazzini [13].

The paper is organized as follows. We review the structure of collinear factorization in the next section. We then present a detailed example calculation, the gluon splitting amplitude with selected external helicities, in section 3. We present the complete set of gluon and fermion splitting amplitudes in section 4, and discuss various checks and comparisons in section 5. In section 6, we use these splitting amplitudes to obtain a function which interpolates smoothly between the collinear and soft limits.

2. The Structure of Collinear Limits

The properties of gauge theories are easiest to discuss in the context of a color decomposition [14]. At tree level, for all-gluon amplitudes such a decomposition takes the form,

$$\mathcal{A}_n^{\text{tree}}(\{k_i, \lambda_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{\lambda_1}, \dots, n^{\lambda_n})), \quad (2.1)$$

where S_n/Z_n is the group of non-cyclic permutations on n symbols, j^{λ_j} denotes the j -th momentum and helicity, and g is the gauge coupling. We use the normalization $\text{Tr}(T^a T^b) = \delta^{ab}$ for the generators of $SU(N)$. One can write analogous formulæ for amplitudes with quark-antiquark pairs or uncolored external lines. The color-ordered or partial amplitude A_n is gauge invariant.

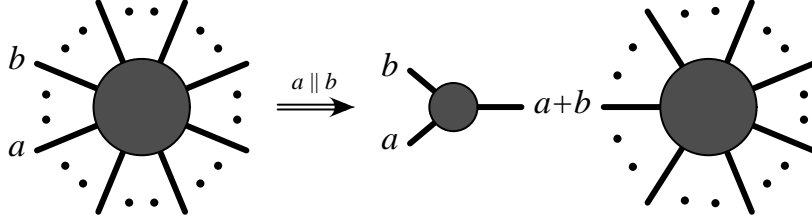


Figure 1. A schematic depiction of the collinear factorization of tree-level amplitudes, with the amplitudes labelled clockwise.

In the collinear limit, $k_a \parallel k_b$ of two adjacent legs, the color-ordered amplitude A_n is singular. (It is finite when the two collinear legs are not adjacent arguments.) This singular behavior has a universal form expressed by the tree-level factorization equation,

$$A_n^{\text{tree}}(1, \dots, a^{\lambda_a}, b^{\lambda_b}, \dots, n) \xrightarrow{k_a \cdot k_b \rightarrow 0} \sum_{\text{ph. pol. } \sigma} \text{Split}_{-\sigma}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(1, \dots, (a+b)^{\sigma}, \dots, n), \quad (2.2)$$

dropping terms finite in the limit. In this equation, $\text{Split}^{\text{tree}}$ is the usual tree splitting amplitude, and the notation ‘ $a+b$ ’ means $k_a + k_b$. The notation ‘ph. pol.’ indicates a sum over physical polarizations only. (‘Physical’ here is in the sense of ‘transverse’, and their number may depend on the number of dimensions and on the variant of dimensional regularization employed.) This factorization is depicted schematically in fig. 1. It is characteristic of gauge theories that the splitting amplitude has a square-root singularity, $\text{Split} \sim 1/\sqrt{s_{ab}}$, rather than a full inverse power of the two-particle invariant.

At one loop, the analogous color decomposition to (2.1) is

$$\mathcal{A}_n(\{k_i, \lambda_i, a_i\}) = g^n \sum_J n_J \sum_{c=1}^{\lfloor n/2 \rfloor + 1} \sum_{\sigma \in S_n/S_{n;c}} \text{Gr}_{n;c}(\sigma) A_{n;c}^{[J]}(\sigma), \quad (2.3)$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x and n_J is the number of particles of spin J . The leading color-structure factor,

$$\text{Gr}_{n;1}(1) = N_c \text{Tr}(T^{a_1} \dots T^{a_n}), \quad (2.4)$$

is just N_c times the tree color factor, and the subleading color structures are given by

$$\text{Gr}_{n;c}(1) = \text{Tr}(T^{a_1} \dots T^{a_{c-1}}) \text{Tr}(T^{a_c} \dots T^{a_n}). \quad (2.5)$$

S_n is the set of all permutations of n objects, and $S_{n;c}$ is the subset leaving $\text{Gr}_{n;c}$ invariant. The decomposition (2.3) holds separately for different spins circulating around the loop. The usual normalization conventions take each massless spin- J particle to have two (helicity) states: gauge bosons, Weyl fermions, and complex scalars. (For internal particles in the fundamental $(N_c + \bar{N}_c)$ representation, only the single-trace color structure ($c = 1$) would be present, and the corresponding color factor would be smaller by a factor of N_c .)

The subleading-color amplitudes $A_{n;c>1}$ are in fact not independent of the leading-color amplitude $A_{n;1}$. Rather, they can be expressed as sums over permutations of the arguments of the latter [5]. (For amplitudes with external fermions, the basic objects are primitive amplitudes [6] rather than the leading-color one, but the same dependence of the subleading color amplitudes holds.)

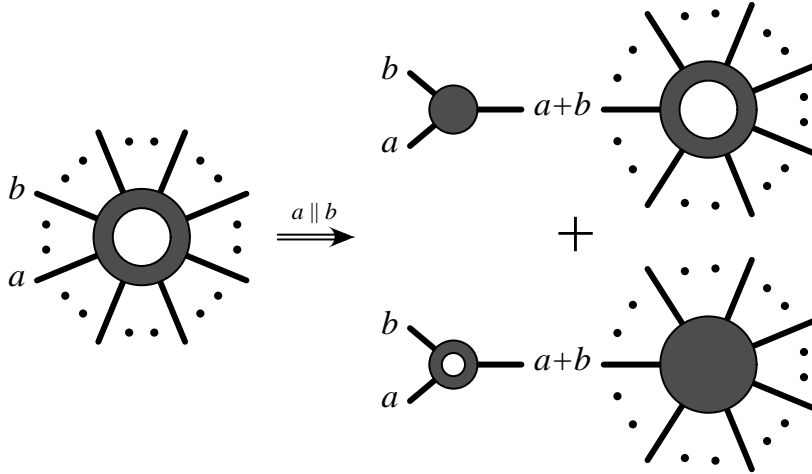


Figure 2. A schematic depiction of the collinear factorization of one-loop amplitudes.

As a result, it suffices to examine the collinear limits of leading-color amplitudes. The collinear limits of the subleading-color then follow using this relation. The leading-color amplitudes obey the following factorization [5,11],

$$A_n^{1\text{-loop}}(1, \dots, a^{\lambda_a}, b^{\lambda_b}, \dots, n) \xrightarrow{a \parallel b} \sum_{\text{ph. pol. } \sigma} \left(\text{Split}_{-\sigma}^{\text{tree}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{1\text{-loop}}(1, \dots, (a+b)^\sigma, \dots, n) + \text{Split}_{-\sigma}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b}) A_{n-1}^{\text{tree}}(1, \dots, (a+b)^\sigma, \dots, n) \right). \quad (2.6)$$

This factorization is depicted schematically in fig. 2.

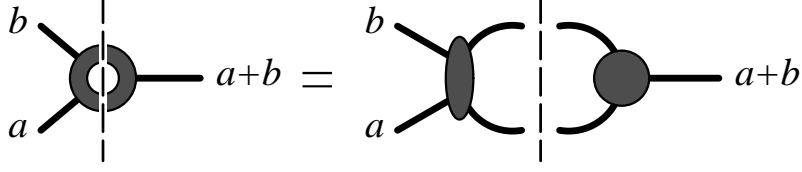


Figure 3. The defining equation for the one-loop splitting amplitude.

This form was originally deduced from explicit calculations of higher-point amplitudes, but can also be proven more generally using the unitarity-based method. The latter proof also provides an explicit formula for the one-loop splitting amplitude [11],

$$\text{Split}_{-\sigma}^{1\text{-loop}}(a^{\lambda_a}, b^{\lambda_b}) = \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}_{-\sigma}^{\text{tree}}((\ell + a + b)^{-\lambda_2}, (-\ell)^{-\lambda_1}) \frac{i}{(\ell + k_a + k_b)^2} \times A_4^{\text{tree}}((-\ell - a - b)^{\lambda_2}, a^{\lambda_a}, b^{\lambda_b}, \ell^{\lambda_1}) \quad (2.7)$$

The restriction to physical polarizations is important; it will give rise to transverse projection operators inside the loop. The defining equation is depicted graphically in fig. 3.

3. A Sample Calculation

As an example of a calculation using eqn. (2.7), we present the calculation of the splitting amplitudes $\text{Split}^{1\text{-loop}}(a^+, b^-)$ in the conventional dimensional regularization (CDR) scheme, but restricted to physical external helicities. (The answer in the original 't Hooft–Veltman (HV) scheme would be identical. For a discussion of the differences of the various schemes, see refs. [15,16].)

For this purpose, we need the three-gluon tree-level splitting amplitude,

$$\text{Split}^{\text{tree}}(P \rightarrow ab; z) = -\frac{\sqrt{2}}{s_{ab}} (-\varepsilon_a \cdot \varepsilon_b k_b \cdot \varepsilon_P + k_b \cdot \varepsilon_a \varepsilon_P \cdot \varepsilon_b - k_a \cdot \varepsilon_b \varepsilon_a \cdot \varepsilon_P) . \quad (3.1)$$

The variable z denotes the collinear momentum fraction, given in the collinear limit by

$$\frac{q \cdot k_a}{q \cdot (k_a + k_b)} \quad (3.2)$$

where q is a *reference* null momentum which is not collinear with k_a or k_b . All momenta are taken to be outgoing, $P + k_a + k_b = 0$. The splitting amplitudes will in general have not only an explicit dependence on the argument z , but also an implicit one that arises from the dependence on the momenta and polarization vectors of legs a and b . Note that the order of the two arguments a and b is important, as the definition of z depends on it. We also need the four-point amplitude $A_4(1^+, 2^-, 3, 4)$,

$$A_4(1^+, 2^-, 3, 4) = \frac{i}{2s_{14}} \left[\langle 2^- | \not{\epsilon}_3 | 1^- \rangle \langle 2^- | \not{\epsilon}_4 | 1^- \rangle - \frac{\langle 2^- | \not{k}_4 | 1^- \rangle \langle 2^- | \not{\epsilon}_4 \not{\epsilon}_3 | 2^+ \rangle}{\langle 1 2 \rangle} + \frac{\langle 2^- | \not{k}_4 | 1^- \rangle \langle 1^+ | \not{\epsilon}_4 \not{\epsilon}_3 | 1^- \rangle}{[1 2]} - \frac{2\varepsilon_3 \cdot \varepsilon_4 \langle 2^- | \not{k}_4 | 1^- \rangle^2}{s_{12}} \right] \quad (3.3)$$

where the notation indicates that the polarization vectors for the last two legs are kept generic, and in $D = 4 - 2\epsilon$ dimensions. In this formula, $s_{ij} = (k_i + k_j)^2$, and following the notation of Xu, Zhang, and Chang [17], $|j^\pm\rangle$ and $\langle j^\pm|$ are massless two-component spinors carrying momentum k_j ; $\langle j l \rangle \equiv \langle j^- | l^+ \rangle$ and $[j l] \equiv \langle j^+ | l^- \rangle$ are spinor products [18,17].

In the CDR scheme, there are $2 - 2\epsilon$ ‘physical’ polarizations, reflected in the identity

$$\sum_{\text{ph. pol. } \sigma} \varepsilon_\mu^\sigma \varepsilon_\nu^{-\sigma} = -g_{\mu\nu} + \frac{q_\mu k_\nu + k_\mu q_\nu}{q \cdot k}, \quad (3.4)$$

where k is the momentum of the gluon and q is the above-mentioned reference momentum.

Using this identity, along with collinear identities such as $\langle b^- | q | a^- \rangle = -\sqrt{z(1-z)} 2q \cdot P$, commuting gamma matrices appropriately, and dropping terms which are finite in the collinear limit, we find for the integrand of eqn. (2.7) a sum of terms,

$$\left[T_0(\ell) - T_0(P - \ell) + T_1(\ell) - T_1(P - \ell) + T_2(\ell) \right] \frac{1}{\ell^2(\ell + k_a + k_b)^2}, \quad (3.5)$$

where

$$\begin{aligned} T_0(p) &= \frac{1}{\sqrt{2}} \sqrt{z(1-z)} q \cdot P \langle b^- | \not{\epsilon}_P | a^- \rangle \frac{1}{q \cdot p s_{a\ell}}, \\ T_1(p) &= \sqrt{2z(1-z)} q \cdot P \frac{p \cdot P}{q \cdot p s_{a\ell}} - \frac{q \cdot P \langle b^- | \not{\epsilon}_P | a^- \rangle \langle b^- | \not{p} | a^- \rangle}{\sqrt{2} s_{ab} q \cdot p s_{a\ell}}, \\ T_2(p) &= -2\sqrt{2} \frac{(1-\epsilon) p \cdot P \langle b^- | \not{p} | a^- \rangle^2}{s_{ab}^2 s_{a\ell}} - 2\sqrt{2} \frac{\langle b^- | \not{\epsilon}_P | a^- \rangle \langle b^- | \not{p} | a^- \rangle}{s_{ab} s_{a\ell}}. \end{aligned} \quad (3.6)$$

The integrals we must perform then include various standard tensor two- and three-point integrals, along with integrals of the following form,

$$\begin{aligned} J_1(s_{ab}, z) &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p - k_a)^2(p - k_a - k_b)^2 p \cdot q}, \\ J_2(s_{ab}, z) &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p - k_a - k_b)^2 p \cdot q}, \\ J_3^\mu(s_{ab}, z) &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu}{p^2(p - k_a)^2(p - k_a - k_b)^2 p \cdot q}, \end{aligned} \quad (3.7)$$

These integrals resemble those of axial gauge. We emphasize however that no explicit choice of gauge has been made above, and because all calculations leading to the cut expression are done on-shell (albeit in $4 - 2\epsilon$ dimensions), none is needed. Furthermore, the splitting amplitudes will be independent of q (as they must be).

The integrals clearly scale like $1/r$ if we scale $q \rightarrow rq$. They can depend only on $q \cdot k_a$ and $q \cdot k_b$ in addition to s_{ab} . Other than the dot products of q with external momenta, however, s_{ab} is the only dimensionful parameter. Furthermore, $q \cdot k_a = z q \cdot (k_a + k_b)$ in the collinear limit, so that the first two integrals necessarily have the form

$$\begin{aligned} J_1(s_{ab}, z) &= \frac{1}{(-s_{ab})^{1+\epsilon} q \cdot (a+b)} f_1(z), \\ J_2(s_{ab}, z) &= \frac{1}{(-s_{ab})^\epsilon q \cdot (a+b)} f_2(z). \end{aligned} \quad (3.8)$$

Consider first J_1 . To evaluate this integral, introduce Feynman parameters, and perform the loop integration,

$$\begin{aligned} J_1(s_{ab}, z) &= -12i \int d^4 c \delta(1 - \Sigma_i c_i) \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{[-(1-c_4)p^2 + 2p \cdot (c_2 k_a + c_3 k_a + c_3 k_b - c_4 q) - c_3 s_{ab}]^4} \\ &= 2 \frac{\Gamma(2+\epsilon)}{(4\pi)^{2-\epsilon}} \int d^4 c \delta(1 - \Sigma_i c_i) \frac{(1-c_4)^{2\epsilon}}{[-c_1 c_3 s_{ab} - 2c_4 q \cdot (k_a + k_b) (c_2 z + c_3)]^{2+\epsilon}} \end{aligned} \quad (3.9)$$

In order to solve for $f_1(z)$, we may set and $q \cdot (k_a + k_b)/s_{ab}$ to any convenient non-zero value. It will be convenient to choose it to be $1/2$; at this point,

$$\begin{aligned} f_1(z) &= -\frac{1}{2}(-s_{ab})^{2+\epsilon} J_1(s_{ab}, z) \Big|_{2q \cdot (k_a + k_b) = s_{ab}} \\ &= -\frac{\Gamma(2+\epsilon)}{(4\pi)^{2-\epsilon}} \int d^4 c \delta(1 - \Sigma_i c_i) \frac{(1-c_4)^{2\epsilon}}{[c_1 c_3 + c_2 c_4 z + c_3 c_4]^{2+\epsilon}}. \end{aligned} \quad (3.10)$$

Using the following change of variables,

$$c_1 = y(1-x), \quad c_2 = (1-y)w, \quad c_3 = (1-y)(1-w), \quad c_4 = yx, \quad (3.11)$$

(with jacobian $y(1-y)$), we find

$$\begin{aligned} f_1(z) &= -\frac{\Gamma(2+\epsilon)}{(4\pi)^{2-\epsilon}} \int_0^1 dx dy dw [y(1-y)]^{-1-\epsilon} \frac{(1-xy)^{2\epsilon}}{[1-w+xwz]^{2+\epsilon}} \\ &= \frac{2}{\epsilon^2} c_\Gamma \left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon) z^{-1-\epsilon}(1-z)^\epsilon - \frac{1}{z} + \frac{(1-z)^\epsilon}{z} {}_2F_1(\epsilon, \epsilon; 1+\epsilon; z) \right] \\ &= \frac{2}{\epsilon^2} c_\Gamma \left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon) z^{-1-\epsilon}(1-z)^\epsilon - \frac{1}{z} + \frac{(1-z)^\epsilon}{z} + \frac{\epsilon^2}{z} \text{Li}_2(z) \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (3.12)$$

where ${}_2F_1$ is the Gauss hypergeometric function, Li_2 the dilogarithm, and the standard one-loop prefactor is,

$$c_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}. \quad (3.13)$$

The other integral in eqn. (3.7), J_2 , is manifestly z -independent, and one finds that

$$f_2(z) = -\frac{1}{\epsilon^2} c_\Gamma. \quad (3.14)$$

The last integral we need can be obtained by a Passarino-Veltman reduction [19,20],

$$J_3^\mu(s_{ab}, z) = \frac{1}{2(-s_{ab})^{1+\epsilon} q \cdot (a+b)} \left[f_1(z) k_a^\mu + \frac{1}{1-z} (2f_2 - z f_1(z)) k_b^\mu + \frac{1}{1-z} (f_1(z) - 2f_2) q^\mu \right] \quad (3.15)$$

With these results, we obtain immediately

$$\begin{aligned} \text{Split}^{1\text{-loop}}(P \rightarrow a^+ b^-; z) &= \\ &= -\frac{1}{2\sqrt{2}s_{ab}} [z f_1(z) + (1-z)f_1(1-z) - 2f_2] \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon \left[\frac{(1-2z)}{\sqrt{z(1-z)}} \langle b^- | \not{\epsilon}_P | a^- \rangle + (k_a - k_b) \cdot \varepsilon_P \right]. \end{aligned} \quad (3.16)$$

The corresponding tree-level result is,

$$-\frac{1}{\sqrt{2}s_{ab}} \left[\frac{(1-2z)}{\sqrt{z(1-z)}} \langle b^- | \not{\epsilon}_P | a^- \rangle + (k_a - k_b) \cdot \epsilon_P \right]. \quad (3.17)$$

4. General Form of Splitting Amplitudes

We can also use the basic eqn. (2.7) for the one-loop splitting amplitudes in order to obtain a more general expression. We can repeat the steps in the previous section to obtain an expression for the gluon splitting amplitude with arbitrary external polarization vectors. In this case, we would use the general four-point amplitude rather than the expression (3.3), sum over polarizations crossing the cut using the identity (3.4), and perform the loop integrals. Doing so, we find (in the CDR scheme) the following result for the one-loop splitting amplitude,

$$\begin{aligned} \text{Split}^{\text{1-loop}}(P \rightarrow a b; z) = & \frac{1}{2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon [z f_1(z) + (1-z) f_1(1-z) - 2 f_2] \text{Split}^{\text{tree}}(P \rightarrow a b; z) \\ & + \frac{1}{\sqrt{2}s_{ab}^2} \frac{\epsilon^2}{(1-2\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2 (k_a - k_b) \cdot \epsilon_P (s_{ab} \epsilon_a \cdot \epsilon_b - 2 k_b \cdot \epsilon_a k_a \cdot \epsilon_b). \end{aligned} \quad (4.1)$$

This calculation requires some tensor integrals beyond those presented in section 3; they are listed in appendix I. As expected, the coefficient proportional to the tree-level structure has $1/\epsilon^2$ and $1/\epsilon$ poles corresponding to soft and collinear singularities of the virtual gluons, as well as an ultraviolet singularity (the formula here is given before ultraviolet subtractions). The second polarization structure, which is new at loop level, is in contrast finite when $\epsilon \rightarrow 0$, as it must be. The two coefficients of the two polarization structures will make additional appearances later on, and so it will be convenient to define

$$\begin{aligned} r_1^{[1]}(z) &= \frac{1}{2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon [z f_1(z) + (1-z) f_1(1-z) - 2 f_2], \\ r_2(z) &= \frac{\epsilon^2}{(1-2\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.2)$$

The first term in eqn. (4.1) is in fact the same in all variants of dimensional regularization, but the second term is not. The corresponding results in the four-dimensional helicity scheme (FDH) [15] may be obtained by correcting for the difference (2 versus $2-2\epsilon$) of helicity states with respect to the CDR scheme. This difference can also be expressed in terms of the contribution of an adjoint scalar inside the loop; for the general result, we find

$$r_2^{[1]}(z) = \frac{(1-\epsilon\delta)\epsilon^2}{(1-2\epsilon)(1-\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2, \quad (4.3)$$

where

$$\delta = \begin{cases} 0 & \text{in the FDH scheme,} \\ 1 & \text{in the CDR (or HV) scheme.} \end{cases}$$

We may write the separate contributions with particles of spin J circulating using the same polarization tensors,

$$\delta \text{Split}_{[J]}^{1\text{-loop}}(P \rightarrow a b; z) = r_1^{[J]}(z) \text{Split}^{\text{tree}}(P \rightarrow a b; z) + r_2^{[J]}(z) \frac{(k_a - k_b) \cdot \varepsilon_P}{\sqrt{2} s_{ab}^2} (s_{ab} \varepsilon_a \cdot \varepsilon_b - 2 k_b \cdot \varepsilon_a k_a \cdot \varepsilon_b) . \quad (4.4)$$

(The formulæ assume two helicity states, that is Weyl or Majorana fermions and complex scalars.)

If we now take the external legs to be in four dimensions, we can write down expressions for the various helicity choices. For example, the following relations hold,

$$\begin{aligned} \text{Split}^{1\text{-loop}}(P^+ \rightarrow a^\pm b^\mp; z) &= \text{Split}^{\text{tree}}(P^+ \rightarrow a^\pm b^\mp; z) r_1(z), \\ \text{Split}^{1\text{-loop}}(P^- \rightarrow a^+ b^+; z) &= \text{Split}^{\text{tree}}(P^- \rightarrow a^+ b^+; z) (r_1(z) - z(1-z) r_2), \\ \text{Split}^{1\text{-loop}}(P^+ \rightarrow a^+ b^+; z) &= \sqrt{z(1-z)} \frac{[a b]}{\langle a b \rangle^2} r_2. \end{aligned} \quad (4.5)$$

The remaining helicity configurations can be obtained using parity.

For completeness, we also record the contribution of a (complex) adjoint scalar circulating in the loop,

$$\begin{aligned} r_1^{[0]} &= 0, \\ r_2^{[0]} &= \frac{\epsilon^2}{(1-2\epsilon)(1-\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.6)$$

For the scalars in the fundamental representation of the gauge group, these functions must be multiplied by a factor of $1/N$.

The basic eqn. (2.7) also applies to splitting amplitudes with fermions either circulating in the loop, or as external states. To compute the contributions to the gluon splitting amplitude proportional to the number of (massless) flavors n_f , we need the tree-level $g \rightarrow q\bar{q}$ splitting amplitude (to the left of the cut), and the two-quark two-gluon amplitude (to the right of the cut),

$$\delta \text{Split}_{[1/2]}^{1\text{-loop}}(P \rightarrow a b; z) = \sum_{\text{ph. pol.}} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}^{\text{tree}}(P \rightarrow (\ell + a + b)_f (-\ell)_f) \frac{i}{(\ell + k_a + k_b)^2} \times A_4^{\text{tree}}((-\ell - a - b)_f, a, b, \ell_f) \quad (4.7)$$

where the f subscript indicates fermionic legs. Evaluating this expression, we obtain for the coefficient of the two tensor structures,

$$\begin{aligned} r_1^{[1/2]} &= 0, \\ r_2^{[1/2]} &= -\frac{\epsilon^2}{(1-2\epsilon)(1-\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.8)$$

The contributions proportional to the number of flavors is thus

$$\delta \text{Split}^{1\text{-loop}}(P \rightarrow a b; z) \Big|_{n_f} = \frac{n_f}{N} \frac{(k_a - k_b) \cdot \varepsilon_P}{\sqrt{2} s_{ab}^2} (s_{ab} \varepsilon_a \cdot \varepsilon_b - 2 k_b \cdot \varepsilon_a k_a \cdot \varepsilon_b) r_2^{[1/2]}. \quad (4.9)$$

The $1/N$ factor reflects the quarks being in the fundamental rather than the adjoint representation.

The one-loop $g \rightarrow q\bar{q}$ splitting amplitude requires, in addition to these tree-level splitting and four-point amplitudes, the four-quark amplitude. Here, there are two contributions which are independent of the number of flavors. One comes from gluons crossing the cut; the other, subleading in the number of colors, from quarks crossing the cut. The former is,

$$\begin{aligned} \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{\text{LC}} = \\ \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}^{\text{tree}}(P \rightarrow (\ell + a + b)(-\ell)) \frac{i}{(\ell + k_a + k_b)^2} \\ \times A_4^{\text{tree}}((-\ell - a - b), a_q, b_{\bar{q}}, \ell). \end{aligned} \quad (4.10)$$

The contribution from quarks crossing the cut is given by the following equation,

$$\begin{aligned} \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{\text{SC}} = \\ \frac{1}{N^2} \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}^{\text{tree}}(P \rightarrow (\ell + a + b)_q(-\ell)_{\bar{q}}) \frac{i}{(\ell + k_a + k_b)^2} \\ \times A_4^{\text{tree}}((-\ell - a - b)_{\bar{q}'}, a_{q'}, b_{\bar{q}}, \ell_q) \end{aligned} \quad (4.11)$$

where the subscript q indicates a quark, and \bar{q} , an antiquark. Although the external quark and antiquark must necessarily have the same flavor, we make use of four-quark amplitudes with two different flavors in order to isolate the n_f -independent contribution. We find for the two different contributions,

$$\begin{aligned} \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{\text{LC}} = \\ \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left\{ r_1^{[1]}(z) - \left[\frac{1}{1-2\epsilon} - \frac{(1-\delta\epsilon)\epsilon}{2(1-\epsilon)} + \frac{2\epsilon(1-\epsilon)(1-\delta\epsilon)}{(1-2\epsilon)(3-2\epsilon)} \right] \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2 \right\} \\ \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{\text{SC}} = -\frac{1}{N^2} \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left(\frac{1}{1-2\epsilon} - \frac{(1-\delta\epsilon)\epsilon}{2(1-\epsilon)} \right) \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2 \end{aligned} \quad (4.12)$$

This splitting amplitude is again scheme-dependent; to compute the difference between the CDR and FDH schemes one can again either modify the gluon sums over polarizations or equivalently compute the contributions of $(-\epsilon)$ complex internal scalars. In the latter method, the Yukawa coupling must be adjusted to obtain the ‘same’ coupling as for a gluon. The adjustment gives precisely the Yukawa coupling that would be obtained in an $N = 4$ supersymmetric gauge theory. The relevant scalar contribution, combining leading- and subleading-color parts, is

$$\begin{aligned} \delta \text{Split}_{[0]}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) = \\ \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left(\frac{\epsilon}{(1-2\epsilon)(3-2\epsilon)} - \frac{\epsilon}{2(1-\epsilon)(1-2\epsilon)} + \frac{1}{N^2} \frac{\epsilon}{2(1-\epsilon)} \right) \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2 \\ = \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left(\frac{\epsilon}{2(1-\epsilon)} - \frac{2\epsilon(1-\epsilon)}{(1-2\epsilon)(3-2\epsilon)} + \frac{1}{N^2} \frac{\epsilon}{2(1-\epsilon)} \right) \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.13)$$

In the first form, the last two terms arise from the Yukawa coupling. The second form matches the scheme-dependent parts shown in eqn. (4.12). In the above equations, the tree splitting amplitude is

$$\text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) = \frac{1}{\sqrt{2}s_{ab}} \bar{u}_a \not{P} u_b. \quad (4.14)$$

Unlike the pure gluon case, for this splitting amplitude no other structure is possible when the fermions are kept on-shell.

The contribution to the $g \rightarrow q\bar{q}$ splitting amplitude proportional to the number of flavors is,

$$\begin{aligned} \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{n_f} &= \frac{n_f}{N} \\ &\times \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}^{\text{tree}}(P \rightarrow (\ell + a + b)_f (-\ell)_f) \frac{i}{(\ell + k_a + k_b)^2} \\ &\times A_4^{\text{tree}}((-\ell - a - b)_f, a_q, b_{\bar{q}}, \ell_f) \\ &= \frac{n_f}{N} \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \frac{2\epsilon(1-\epsilon)}{(1-2\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.15)$$

We turn next to the fermion splitting amplitude, $q \rightarrow qg$. All contributions here involve a tree-level $q \rightarrow qg$ splitting amplitude to the left of the cut, and a two-quark two-gluon amplitude to the right; but there are both leading- and subleading-color ones. The leading-color term is,

$$\begin{aligned} \text{Split}^{1\text{-loop}}(P_q \rightarrow a_q b; z) &= \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}^{\text{tree}}(P_q \rightarrow (\ell + a + b)_f (-\ell)) \frac{i}{(\ell + k_a + k_b)^2} \\ &\times A_4^{\text{tree}}((-\ell - a - b)_f, a_q, b, \ell). \end{aligned} \quad (4.16)$$

The result can be written in terms of the tree-level splitting amplitude, along with a new polarization tensor,

$$\begin{aligned} \text{Split}^{\text{tree}}(P_q \rightarrow a_q b; z) &= -\frac{1}{\sqrt{2}s_{ab}} \bar{u}_a \not{P} u_{-P}, \\ \text{Split}^{1\text{-loop}}(P_q \rightarrow a_q b; z) &= \text{Split}^{\text{tree}}(P_q \rightarrow a_q b; z) r_3^{\text{LC}} - \frac{\sqrt{2}}{s_{ab}^2} \bar{u}_a \not{k}_b u_{-P} k_a \cdot \varepsilon_b r_4^{\text{LC}}. \end{aligned} \quad (4.17)$$

Note that the momentum P is outgoing (in accord with our overall convention), and hence u_{-P} is the usual spinor for an inward-directed momentum. Given our conventions, helicity conservation means that the two external fermions at the ends of a given fermion line have *opposite* helicities: flipping the direction of a momentum also flips the sign of the helicity.

The functions $r_{3,4}$ appearing in eqn. (4.17) are,

$$\begin{aligned} r_3^{\text{LC}} &= \frac{1}{2} \left((1-z)f_1(1-z) - \frac{(1-\delta\epsilon)\epsilon^2}{(1-\epsilon)(1-2\epsilon)} f_2 \right) \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon, \\ r_4^{\text{LC}} &= \frac{1}{2} \frac{(1-\delta\epsilon)\epsilon^2}{(1-\epsilon)(1-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.18)$$

The subleading-color terms are given by a formula similar to (4.16), but with the fermion legs not adjacent in the four-point amplitude to the right of the cut,

$$\begin{aligned} \delta \text{Split}^{1\text{-loop}}(P_q \rightarrow a_q b; z) \Big|_{\text{SC}} &= \\ &= -\frac{1}{N^2} \sum_{\text{ph. pol. } \lambda_1, \lambda_2} \int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-2\epsilon}} \frac{i}{\ell^2} \text{Split}^{\text{tree}}(P_q \rightarrow (\ell + a + b)_f (-\ell)) \frac{i}{(\ell + k_a + k_b)^2} \\ &\times A_4^{\text{tree}}((-\ell - a - b)_f, a_q, \ell, b). \end{aligned} \quad (4.19)$$

We may express the result of the subleading-color contributions in a decomposition similar to that in eqn. (4.17), but multiplied by an overall factor of $1/N^2$. The coefficient functions are,

$$\begin{aligned} r_3^{\text{SC}} &= -\frac{1}{2} \left(z f_1(z) - 2f_2 + \frac{(1-\delta\epsilon)\epsilon^2}{(1-\epsilon)(1-2\epsilon)} f_2 \right) \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon, \\ r_4^{\text{SC}} &= \frac{1}{2} \frac{(1-\delta\epsilon)\epsilon^2}{(1-\epsilon)(1-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.20)$$

The difference between schemes again turns out to be expressible using contributions in which the gluon crossing the cut is replaced by a scalar; then for both the leading-color and subleading-color terms, these contributions are,

$$\begin{aligned} r_3^{[0]} &= -\frac{\epsilon^2}{2(1-\epsilon)(1-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2, \\ r_4^{[0]} &= \frac{\epsilon^2}{2(1-\epsilon)(1-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2. \end{aligned} \quad (4.21)$$

(The subleading-color contributions are of course again multiplied by a factor of $1/N^2$, not included here.)

For specific helicity choices for the external legs, we then find,

$$\begin{aligned} \text{Split}^{1\text{-loop}}(P_q^- \rightarrow a_q^+ b^-; z) &= \text{Split}^{\text{tree}}(P_q^- \rightarrow a_q^+ b^-; z)(r_3 + r_4), \\ \text{Split}^{1\text{-loop}}(P_q^- \rightarrow a_q^+ b^+; z) &= \text{Split}^{\text{tree}}(P_q^- \rightarrow a_q^+ b^+; z)(r_3 + z r_4). \end{aligned} \quad (4.22)$$

5. Consistency Checks

The splitting amplitudes computed in the previous section may appear to be completely independent quantities. They are not. If we set $n_f/N \rightarrow 1$ and $1/N^2 \rightarrow -1$, we obtain the corresponding quantities in a supersymmetric theory. These latter satisfy various identities descended from the supersymmetry Ward identity [21]. Such identities thus provide a cross-check on the splitting amplitudes even in nonsupersymmetric theories such as QCD.

Supersymmetry Ward identities have long been recognized [22,1] as providing useful relations between different amplitudes in gauge theories. For example, they relate amplitudes with n gluons to those with $(n-2)$ gluons and two gluinos. For certain choices of the external helicities, these identities, which hold to all orders in perturbation theory, allow us to solve for the two-gluino amplitudes in terms of the n -gluon ones. For example,

$$A_n^{\text{tree}}(1_{\tilde{g}}^+, 2^-, 3^+, \dots, j_{\tilde{g}}^-, \dots, n^+) = -\frac{\langle 12 \rangle}{\langle 2j \rangle} A_n^{\text{tree}}(1^+, 2^-, 3^+, \dots, j^-, \dots, n^+). \quad (5.1)$$

At tree level, the two-gluino color-ordered amplitudes are in fact identical to two-quark ones, so the identity links two different QCD color-ordered amplitudes. Beyond tree level, color-ordered QCD amplitudes are of course no longer identical to those in a supersymmetric theory. Supersymmetry identities are nonetheless still useful in relating different contributions to QCD amplitudes [10].

The splitting amplitudes are in fact identical for all matterless supersymmetric gauge theories, independent of the number of supersymmetries. We may therefore pick an $N=1$ supersymmetric gauge theory, with no matter content, to study the supersymmetry identities.

What relations do we expect for the splitting amplitudes? If we consider the $n = 5$ case of eqn. (5.1), with legs relabeled, we obtain the following relation between an all-gluon and a two-gluino three-gluon amplitude,

$$\langle 31 \rangle A_5(1^+, 2^+, 3^-, 4^+, 5^-) = \langle 35 \rangle A_5(1_{\tilde{g}}^+, 2^+, 3^-, 4^+, 5_{\tilde{g}}^-). \quad (5.2)$$

This relation holds to all orders in perturbation theory. Let us now examine the collinear limit $k_1 \parallel k_5$. At tree-level, we then obtain the identity ($P = -k_1 - k_5$)

$$\begin{aligned} \langle 31 \rangle A^{\text{tree}}(2^+, 3^-, 4^+, (-P)^-) \text{Split}^{\text{tree}}(P^+ \rightarrow 5^- 1^+; z) = \\ \langle 35 \rangle A^{\text{tree}}(2^+, 3^-, 4^+, (-P)^-) \text{Split}^{\text{tree}}(P^+ \rightarrow 5_{\tilde{g}}^- 1_{\tilde{g}}^+; z). \end{aligned} \quad (5.3)$$

(Note that the pure gluon amplitudes with a lone negative helicity vanish to all orders in the supersymmetric theory. The sum over the polarizations of the fused leg $k_1 + k_5$ thus reduces to a single term.) This identity in turn implies a relation between the $g \rightarrow gg$ and $g \rightarrow \tilde{g}\tilde{g}$ splitting amplitudes,

$$\text{Split}^{\text{tree}}(P^+ \rightarrow a_{\tilde{g}}^- b_{\tilde{g}}^+; z) = \sqrt{\frac{1-z}{z}} \text{Split}^{\text{tree}}(P^+ \rightarrow a^- b^+; z), \quad (5.4)$$

where we have used $k_5 = z(k_1 + k_5)$ and $k_1 = (1-z)(k_1 + k_5)$ in the collinear limit to evaluate the ratio $\langle 31 \rangle / \langle 35 \rangle$.

If we now repeat this exercise for the one-loop amplitudes, again taking the limit of eqn. (5.2), we obtain an identity of the same form for the loop splitting amplitudes,

$$\text{Split}^{1\text{-loop}}(P^+ \rightarrow a_{\tilde{g}}^- b_{\tilde{g}}^+; z) = \sqrt{\frac{1-z}{z}} \text{Split}^{1\text{-loop}}(P^+ \rightarrow a^- b^+; z). \quad (5.5)$$

The similarity is quite natural as the Ward identities hold to all orders in perturbation theory.

One subtlety that arises in adapting the computations of the previous section to use in a supersymmetric theory is a difference in phase conventions. It is therefore more convenient to take the ratio of the tree-level and one-loop identities,

$$\frac{\text{Split}^{1\text{-loop}}(P^+ \rightarrow a_{\tilde{g}}^- b_{\tilde{g}}^+; z)}{\text{Split}^{\text{tree}}(P^+ \rightarrow a_{\tilde{g}}^- b_{\tilde{g}}^+; z)} = \frac{\text{Split}^{1\text{-loop}}(P^+ \rightarrow a^- b^+; z)}{\text{Split}^{\text{tree}}(P^+ \rightarrow a^- b^+; z)}, \quad (5.6)$$

as a test of our computation. This form of the identity is not dependent on the relative phase conventions for gluons and gluinos.

In performing this check, we should of course use a supersymmetry-preserving regulator, so we pick the FDH scheme ($\delta = 0$).

To obtain the $g \rightarrow \tilde{g}\tilde{g}$ splitting amplitude, start with the $g \rightarrow q\bar{q}$ amplitude (4.12,4.15),

$$\begin{aligned} \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left[r_1^{[1]}(z) - 2 \frac{(1-\epsilon)(1-n_f/N-\delta\epsilon)\epsilon}{(1-2\epsilon)(3-2\epsilon)} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2 \right. \\ \left. - \left(1 + \frac{1}{N^2} \right) \left[\frac{1}{1-2\epsilon} - \frac{(1-\delta\epsilon)}{2(1-\epsilon)} \right] \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon f_2 \right], \end{aligned} \quad (5.7)$$

and set $n_f/N \rightarrow 1$, $1/N^2 \rightarrow -1$, and $\delta = 0$:

$$\text{Split}^{1\text{-loop}}(P \rightarrow a_{\tilde{g}} b_{\tilde{g}}; z) = \text{Split}^{\text{tree}}(P \rightarrow a_{\tilde{g}} b_{\tilde{g}}; z) r_1^{[1]}(z). \quad (5.8)$$

In an $N = 1$ supersymmetric theory, the fermionic contribution to $g \rightarrow gg$ cancels the $r_2^{[1]}$ term in the gluon contribution, and so we obtain,

$$\text{Split}^{1\text{-loop}}(P \rightarrow a b; z) = \text{Split}^{\text{tree}}(P \rightarrow a b; z) r_1^{[1]}(z). \quad (5.9)$$

Both sides of eqn. (5.6) are thus equal to $r_1^{[1]}(z)$, and the identity is obeyed.

The vanishing of $r_2^{[1]} + r_2^{[1/2]}$ is in fact also a result of a supersymmetry Ward identity. To see this, make use of the following Parke–Taylor equation [23],

$$A_n(1^+, 2^+, \dots, n^-) = 0, \quad (5.10)$$

which are consequences of the Ward identities along with helicity conservation for fermionic amplitudes. Consider the collinear limit $k_1 \parallel k_2$ of this equation at tree level. Using the above equation itself, the polarization sum reduces to a lone term ($P = -(k_1 + k_2)$),

$$A_n^{\text{tree}}((-P)^-, 3^+, \dots, n^-) \text{Split}(P^+ \rightarrow 1^+ 2^+; z) = 0. \quad (5.11)$$

Since the amplitude does not vanish, the splitting amplitude must. Repeating the above limit for loop amplitudes, we see that this result holds to all orders in perturbation theory. But in an $N = 1$ supersymmetric theory, this implies that

$$\text{Split}(P^+ \rightarrow a^+ b^+; z)|_{[1]} + \text{Split}(P^+ \rightarrow a^+ b^+; z)|_{[1/2]} = 0 \quad \text{or} \quad r_2^{[1]} = -r_2^{[1/2]}. \quad (5.12)$$

If we consider theories with additional supersymmetries, or massless matter multiplets in an $N = 1$ theory, we can also relate the contribution of internal scalars to the fermionic contribution:

$$\text{Split}(P^+ \rightarrow a^+ b^+; z)|_{[1/2]} + \text{Split}(P^+ \rightarrow a^+ b^+; z)|_{[0]} = 0 \quad \text{or} \quad r_2^{[1/2]} = -r_2^{[0]}. \quad (5.13)$$

Inspection of eqns. (4.6) and (4.8) shows that the results presented herein indeed satisfy this relation.

We can also derive a relation between the $g \rightarrow gg$ and the $\tilde{g} \rightarrow \tilde{g}g$ splitting amplitudes. To do so consider the collinear limit $k_4 \parallel k_5$ of eqn. (5.2) at tree-level ($P = -k_4 - k_5$),

$$\begin{aligned} \langle 31 \rangle A^{\text{tree}}(1^+, 2^+, 3^-, (-P)^-) \text{Split}^{\text{tree}}(P^+ \rightarrow 4^+ 5^-; z) = \\ \langle 35 \rangle A^{\text{tree}}(1_{\tilde{g}}^+, 2^+, 3^-, (-P)_{\tilde{g}}^-) \text{Split}^{\text{tree}}(P_{\tilde{g}}^+ \rightarrow 4^+ 5_{\tilde{g}}^-; z). \end{aligned} \quad (5.14)$$

Using the relation

$$\langle 31 \rangle A(1^+, 2^+, 3^-, 4^-) = \langle 34 \rangle A(1_{\tilde{g}}^+, 2^+, 3^-, 4_{\tilde{g}}^-), \quad (5.15)$$

we obtain

$$\text{Split}(P_{\tilde{g}}^+ \rightarrow a^+ b_{\tilde{g}}^-; z) = \frac{1}{\sqrt{1-z}} \text{Split}(P^+ \rightarrow a^+ b^-; z). \quad (5.16)$$

We have omitted the superscript ‘tree’ because repeating the limit for loop amplitudes we find that it holds for them as well. To avoid problems with phase conventions, it is again better to form ratios,

$$\frac{\text{Split}^{1\text{-loop}}(P_{\tilde{g}}^+ \rightarrow a^+ b_{\tilde{g}}^-; z)}{\text{Split}^{\text{tree}}(P_{\tilde{g}}^+ \rightarrow a^+ b_{\tilde{g}}^-; z)} = \frac{\text{Split}^{1\text{-loop}}(P^+ \rightarrow a^+ b^-; z)}{\text{Split}^{\text{tree}}(P^+ \rightarrow a^+ b^-; z)}. \quad (5.17)$$

Setting $1/N^2 \rightarrow -1$ in eqns. (4.17,4.18,4.20) we obtain,

$$\text{Split}^{1\text{-loop}}(P_{\tilde{g}} \rightarrow a_{\tilde{g}} b; z) = \text{Split}^{\text{tree}}(P_{\tilde{g}} \rightarrow a_{\tilde{g}} b; z) r_1^{[1]}(z) \quad (5.18)$$

for the $\tilde{g} \rightarrow \tilde{g}g$ splitting amplitude, with $r_1^{[1]}(z)$ defined in eqn. (4.2). As a result of setting $1/N^2 \rightarrow -1$, the function multiplying the structure

$$\overline{u}_a \not{k}_b u_{-P} k_a \cdot \varepsilon_b \quad (5.19)$$

(new at the one-loop level) vanishes. We are thus left with only one function needed to describe the one loop result for the $\tilde{g} \rightarrow \tilde{g}g$ splitting amplitude, independent of the external helicities. We note that although one should use a supersymmetry-preserving regularisation scheme (i.e. FDH) it is remarkable that the difference between the results in the FDH and CDR schemes cancels out in the expression above.

Comparing with eqn. (5.9), we see that this second supersymmetry identity is also satisfied.

Given the observation that in supersymmetric theories, only one universal function multiplying the tree-level results appears in the one-loop splitting amplitudes, the two checks described above for a specific combination of helicities are sufficient to check the supersymmetry identities for all possible helicities.

We may also compare our results with results previously obtained in the literature. In particular, we will compare with the result of Bern, Del Duca, and Schmidt [8] for the $g \rightarrow gg$ splitting amplitude to all orders in ϵ , and the results (to $\mathcal{O}(\epsilon^0)$) for the $g \rightarrow q\bar{q}$ and $q \rightarrow qg$ splitting amplitudes quoted in ref. [5]. For the comparison with ref. [8] it is convenient to use a slightly different representation for $f_1(z)$ than the one given in eqn. (3.10). Transforming the arguments of the hypergeometric function by the use of two identities, eqns. (9.131.1) and (9.132.2) of ref. [24] we may write

$$z f_1(z) = \frac{2}{\epsilon^2} c_\Gamma \left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon) \left(\frac{1-z}{z} \right)^\epsilon - \frac{z}{1-z} \frac{\epsilon}{1+\epsilon} {}_2F_1(1, 1+\epsilon; 2+\epsilon; \frac{z}{z-1}) \right], \quad (5.20)$$

and

$$(1-z)f_1(1-z) = -\frac{2}{\epsilon^2} c_\Gamma {}_2F_1(1, -\epsilon; 1-\epsilon; \frac{z}{z-1}). \quad (5.21)$$

We may now expand the hypergeometric functions in terms of polylogarithms, to obtain

$$-\frac{z}{1-z} \frac{\epsilon}{1+\epsilon} {}_2F_1(1, 1+\epsilon; 2+\epsilon; \frac{z}{z-1}) - {}_2F_1(1, -\epsilon; 1-\epsilon; \frac{z}{z-1}) = 2 \sum_{k=1,3,5,\dots}^{\infty} \epsilon^k \text{Li}_k \left(\frac{-z}{1-z} \right) - 1, \quad (5.22)$$

and thence,

$$r_1^{[1]}(z) = c_\Gamma \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon \left[-\Gamma(1-\epsilon)\Gamma(1+\epsilon) \left(\frac{1-z}{z} \right)^\epsilon + 2 \sum_{k=1,3,5,\dots}^{\infty} \epsilon^k \text{Li}_k \left(\frac{-z}{1-z} \right) \right], \quad (5.23)$$

where the polylogarithms are defined as follows [25],

$$\begin{aligned} \text{Li}_1(x) &= -\ln(1-x), \\ \text{Li}_k(x) &= \int_0^x \frac{dt}{t} \text{Li}_{k-1}(t) \quad (k=2,3,\dots). \end{aligned} \quad (5.24)$$

Comparing eqn. (5) of ref. [8] with eqn. (4.5), we see that comparing our results with theirs reduces to checking the following pair of equations,

$$\begin{aligned} G^n &= r_1^{[1]}(z), \\ G^f &= -z(1-z) \left(r_2^{[1/2]} + r_2^{[1]} \right). \end{aligned} \quad (5.25)$$

Comparing eqn. (5.23), and eqns. (4.3, 4.8) with the results presented in ref. [8], it is easy to see that the one-loop results shown in this paper indeed satisfy eqn. (5.25) and thus agree with the result[†] given by Bern, Del Duca and Schmidt to all orders in the dimensional regularization parameter ϵ . The contribution of an internal scalar is related to the difference between the FDH and CDR schemes in the gluon splitting amplitudes. Since our results agree with ref. [8] for both schemes, we implicitly agree on the scalar contribution as well.

The representation of $r_1^{[1]}$ given in eqn. (5.23) is also useful for comparing the results for the $g \rightarrow q\bar{q}$ splitting amplitudes with the results quoted in the literature [5]. Using

$$\Gamma(1-\epsilon)\Gamma(1+\epsilon) = 1 + \frac{\pi^2}{6}\epsilon^2 + O(\epsilon^4), \quad (5.26)$$

we obtain from eqn. (5.23)

$$\begin{aligned} r_1^{[1]}(z) &= \frac{c_\Gamma}{\epsilon^2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon \left[-1 + \epsilon \ln(1-z) + \epsilon \ln(z) - \frac{\epsilon^2}{2} \ln^2\left(\frac{1-z}{z}\right) - \frac{\pi^2}{6}\epsilon^2 + O(\epsilon^3) \right] \\ &= \frac{c_\Gamma}{\epsilon^2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon \left[1 - \left(\frac{1}{z} \right)^\epsilon - \left(\frac{1}{1-z} \right)^\epsilon + \epsilon^2 \ln(z) \ln(1-z) - \frac{\pi^2}{6}\epsilon^2 + O(\epsilon^3) \right], \end{aligned} \quad (5.27)$$

for the expansion in ϵ . Together with the expansion of the prefactors,

$$\begin{aligned} \frac{2\epsilon(1-\epsilon)(1-\delta\epsilon)}{(1-2\epsilon)(3-2\epsilon)} &= \frac{2}{3}\epsilon + \left(\frac{10}{9} - \frac{2}{3}\delta \right) \epsilon^2 + O(\epsilon^3), \\ \frac{1}{1-2\epsilon} - \frac{(1-\delta\epsilon)\epsilon}{2(1-\epsilon)} &= 1 + \frac{3}{2}\epsilon + \left(\frac{7}{2} + \frac{1}{2}\delta \right) \epsilon^2 + O(\epsilon^3), \\ \frac{\epsilon}{(1-2\epsilon)(3-2\epsilon)} &= \frac{1}{3}\epsilon + \frac{8}{9}\epsilon^2 + O(\epsilon^3), \end{aligned} \quad (5.28)$$

appearing in eqn. (4.12), (4.15) and (4.13), this leads immediately to the following expansions in ϵ for the $g \rightarrow q\bar{q}$ splitting amplitudes,

$$\begin{aligned} \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{n_f} &= \frac{n_f}{N} c_\Gamma \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left[-\frac{2}{3} \frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon - \frac{10}{9} - O(\epsilon) \right] \\ \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{\text{SC}} &= -\frac{1}{N^2} c_\Gamma \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \\ &\quad \left[-\frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon - \frac{3}{2} \frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon - \left(\frac{7}{2} + \frac{1}{2}\delta \right) + O(\epsilon) \right] \\ \delta \text{Split}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) \Big|_{\text{LC}} &= c_\Gamma \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \\ &\quad \left\{ -\frac{1}{\epsilon^2} \left[\left(\frac{\mu^2}{z(-s_{ab})} \right)^\epsilon + \left(\frac{\mu^2}{(1-z)(-s_{ab})} \right)^\epsilon - 2 \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon \right] \right. \\ &\quad \left. + \frac{13}{6} \frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon + \ln(z) \ln(1-z) - \frac{\pi^2}{6} + \frac{83}{18} - \frac{1}{6}\delta + O(\epsilon) \right\} \end{aligned} \quad (5.29)$$

[†] Note that there is misprint in eqn. (9) of the journal version of ref. [8] due to a misplaced line break. One should replace $\Gamma \times (1+\epsilon)$ by $\Gamma(1+\epsilon)$.

For the contribution from virtual scalars we obtain,

$$\delta \text{Split}_{[0]}^{1\text{-loop}}(P \rightarrow a_q b_{\bar{q}}; z) = c_\Gamma \text{Split}^{\text{tree}}(P \rightarrow a_q b_{\bar{q}}; z) \left[-\frac{1}{3} \frac{1}{\epsilon} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon - \frac{8}{9} + \mathcal{O}(\epsilon) \right], \quad (5.30)$$

if we consider only the first term in eqn. (4.13). (Recall that the two other terms in eqn. (4.13) are due to the coupling of the scalars to quarks.) These expansions in ϵ agree with the results for the $g \rightarrow q\bar{q}$ splitting amplitudes presented in eqn. (B.12) of ref. [5].

We close these comparisons with that for the $q \rightarrow qg$ splitting amplitude. For this purpose, we must expand the functions $r_3^{\text{LC,SC}}, r_4^{\text{LC,SC}}$ in ϵ . This can be easily done using the following expression for the function $zf_1(z)$,

$$\begin{aligned} zf_1(z) &= \frac{2}{\epsilon^2} c_\Gamma \left[-\left(\frac{1}{z}\right)^\epsilon + \epsilon^2 \left(\text{Li}_2(z) + \ln(z) \ln(1-z) - \frac{\pi^2}{6} \right) \right] + \mathcal{O}(\epsilon) \\ &= \frac{2}{\epsilon^2} c_\Gamma \left[-\left(\frac{1}{z}\right)^\epsilon - \epsilon^2 \text{Li}_2(1-z) \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (5.31)$$

where we have used the dilogarithm identity

$$\text{Li}_2(x) + \text{Li}_2(1-x) = \frac{\pi^2}{6} - \ln(x) \ln(1-x). \quad (5.32)$$

The connection between the functions $r_3^{\text{LC,SC}}, r_4^{\text{LC,SC}}$ used in this paper and the functions $r_S(q^+, a^-)$, $r_S(q^+, a^+)$ introduced in eqn. (B.7) of ref. [5] can be read off from eqn. (4.22):

$$\begin{aligned} c_\Gamma r_S(q^+, a^-) &= r_3^{\text{LC}} + r_4^{\text{LC}} + \frac{1}{N^2} (r_3^{\text{SC}} + r_4^{\text{SC}}) \\ c_\Gamma r_S(q^+, a^+) &= r_3^{\text{LC}} + r_4^{\text{LC}} + \frac{1}{N^2} (r_3^{\text{SC}} + r_4^{\text{SC}}) - (1-z) \left(r_4^{\text{LC}} + \frac{r_4^{\text{SC}}}{N^2} \right) \end{aligned} \quad (5.33)$$

Using the expansion for $zf_1(z)$ we obtain

$$\begin{aligned} r_3^{\text{LC}} + r_4^{\text{LC}} + \frac{1}{N^2} (r_3^{\text{SC}} + r_4^{\text{SC}}) &= c_\Gamma \left[-\frac{1}{\epsilon^2} \left(\frac{\mu^2}{(1-z)(-s_{ab})} \right)^\epsilon - \text{Li}_2(z) \right] \\ &\quad - c_\Gamma \frac{1}{N^2} \left[-\frac{1}{\epsilon^2} \left(\frac{\mu^2}{z(-s_{ab})} \right)^\epsilon + \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{ab}} \right)^\epsilon - \text{Li}_2(1-z) \right] + \mathcal{O}(\epsilon), \\ -(1-z) \left(r_4^{\text{LC}} + \frac{1}{N^2} r_4^{\text{SC}} \right) &= c_\Gamma (1 + \frac{1}{N^2}) \frac{1}{2} (1-z) + \mathcal{O}(\epsilon). \end{aligned} \quad (5.34)$$

These results agree with eqns. (B.10) and (B.11) of ref. [5].

6. The One-Loop Antenna Function

Gauge-theory amplitudes are also singular in the soft limit, when a gluon four-momentum vanishes. Color-ordered amplitudes display a factorization similar to that in eqn. (2.6),

$$\begin{aligned} A_n^{1\text{-loop}}(1, \dots, a, s^{\lambda_s}, b, \dots, n) &\xrightarrow{k_s \rightarrow 0} \\ &\text{Soft}^{\text{tree}}(a, s^{\lambda_s}, b) A_{n-1}^{1\text{-loop}}(1, \dots, a, b, \dots, n) \\ &\quad + \text{Soft}^{1\text{-loop}}(a, s^{\lambda_s}, b) A_{n-1}^{\text{tree}}(1, \dots, a, b, \dots, n), \end{aligned} \quad (6.1)$$

in which the collinear splitting amplitude *Split* is replaced by the eikonal amplitude *Soft*. The eikonal limit also demonstrates one of the advantages of the looking at color-ordered amplitudes, namely that they allow a simple and straightforward factorization. The amplitude as a whole does not factorize simply, because color-charge factors get tangled up with the eikonal function. The eikonal function may be extracted to $\mathcal{O}(\epsilon)$ from known results for five-gluon amplitudes. It has also been computed to all orders in ϵ by Bern, Del Duca, and Schmidt [8].

In some applications, for example the construction of jet programs, one wishes to integrate over the singular regions. One can do this by integrating over the collinear and soft regions, and using in each the factorization appropriate to it. However, one must then introduce an artificial boundary between the two. It would be nicer to have a single, unifying function which summarizes the behavior in all singular regions.

Catani and Seymour [26] wrote down just such a function, which appears in a *dipole* factorization formula. They presented a single function capturing the singular behavior of the squared matrix element in both the soft and collinear limits. One of us then gave a derivation of a single *antenna* factorization function [27] — so-called because it is within each color antenna that the two limits are combined — at the amplitude level. In this section, we will construct the one-loop analog of the antenna function and the antenna factorization formula.

The antenna function presented in ref. [27] describes the factorization of an amplitude when a color-adjacent trio of legs a , 1, and b degenerates into two independent, hard momenta \hat{a} and \hat{b} , without a or b becoming soft. (This is the limit where $\Delta(a, 1, b)/s_{ab}^3 \rightarrow 0$, Δ being the symmetric Gram determinant of the $(a, 1, b)$ system.) At tree level, the antenna function is given by the following formula,

$$\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, b) = \frac{1}{s_{a1}} \varepsilon_{\hat{b}} \cdot \varepsilon_b V_3(a, 1, \hat{a}) + \frac{1}{s_{b1}} \varepsilon_{\hat{a}} \cdot \varepsilon_a V_3(1, b, \hat{b}), \quad (6.2)$$

where V_3 is the color-ordered gluon three-point vertex. In addition, we must specify a pair of *reconstruction* functions $k_{\hat{a}, \hat{b}} = f_{\hat{a}, \hat{b}}(k_a, k_1, k_b)$ such that $(k_{\hat{a}}, k_{\hat{b}}) \rightarrow -(k_a + k_1, k_b)$ sufficiently quickly when $k_1 \parallel k_a$, and likewise $(k_{\hat{a}}, k_{\hat{b}}) \rightarrow -(k_a, k_b + k_1)$ sufficiently quickly when $k_1 \parallel k_b$. These functions are not unique; different choices will lead to functional forms for the antenna function that differ by terms non-singular in any of the singular limits. We can also express the tree-level antenna function in terms of the tree-level splitting amplitude as follows,

$$\text{Ant}^{\text{tree}}(\hat{a}, \hat{b} \leftarrow a, 1, b) = -\varepsilon_{\hat{b}} \cdot \varepsilon_b \text{Split}^{\text{tree}}(\hat{a} \rightarrow a \, 1; 1 - \frac{s_{1b}}{K^2}) - \varepsilon_{\hat{a}} \cdot \varepsilon_a \text{Split}^{\text{tree}}(\hat{b} \rightarrow 1 \, b; \frac{s_{a1}}{K^2}), \quad (6.3)$$

where $K^2 = s_{\hat{a}\hat{b}} = (k_a + k_1 + k_b)^2$.

By analogy, we may now define the following function,

$$L(\hat{a}, \hat{b} \leftarrow a, 1, b) = -\varepsilon_{\hat{b}} \cdot \varepsilon_b \text{Split}^{1\text{-loop}}(\hat{a} \rightarrow a \, 1; 1 - \frac{s_{1b}}{K^2}) - \varepsilon_{\hat{a}} \cdot \varepsilon_a \text{Split}^{1\text{-loop}}(\hat{b} \rightarrow 1 \, b; \frac{s_{a1}}{K^2}). \quad (6.4)$$

What are the limits of this function in the three singular regions, $k_a \parallel k_1$, $k_b \parallel k_1$, and $k_1 \rightarrow 0$?

In the first region, $\text{Split}^{1\text{-loop}}(\hat{b} \rightarrow 1 \, b; z)$ has at worst a logarithmic singularity as $s_{a1} \rightarrow 0$. Furthermore, k_b can be thought of as a reference momentum for the purposes of defining z ,

$$z = \frac{s_{ab}}{s_{1b} + s_{ab}} \simeq \frac{s_{ab}}{s_{a1} + s_{1b} + s_{ab}}. \quad (6.5)$$

The L function thus reduces to the ordinary one-loop splitting amplitude in this region. Similarly, it reduces to the splitting amplitude in the region $k_b \parallel k_1$.

In the soft region, $k_1 \rightarrow 0$,

$$\begin{aligned}\text{Split}^{\text{tree}}(\hat{a} \rightarrow a\,1; z) &\rightarrow \frac{\sqrt{2}}{s_{a1}} k_a \cdot \varepsilon_1 \varepsilon_a \cdot \varepsilon_{\hat{a}}, \\ \text{Split}^{\text{tree}}(\hat{b} \rightarrow 1\,b; z) &\rightarrow -\frac{\sqrt{2}}{s_{1b}} k_b \cdot \varepsilon_1 \varepsilon_b \cdot \varepsilon_{\hat{b}},\end{aligned}\tag{6.6}$$

while the second term in eqn. (4.1) is finite, and thus may be dropped. In taking the limit of L , we may set the relevant z to either zero or one as appropriate, except in expressions of the form z^ϵ . The leading behavior of L in the soft limit is thus given by

$$\begin{aligned}-\frac{1}{2} \left\{ \left(\frac{\mu^2}{-s_{a1}} \right)^\epsilon \left[\frac{s_{1b}}{K^2} f_1 \left(\frac{s_{1b}}{K^2} \right) + f_1 \left(1 - \frac{s_{1b}}{K^2} \right) - 2f_2 \right] \frac{\sqrt{2}}{s_{a1}} k_a \cdot \varepsilon_1 \varepsilon_a \cdot \varepsilon_{\hat{a}} \varepsilon_b \cdot \varepsilon_{\hat{b}} \right. \\ \left. - \left(\frac{\mu^2}{-s_{1b}} \right)^\epsilon \left[\frac{s_{1a}}{K^2} f_1 \left(\frac{s_{1a}}{K^2} \right) + f_1 \left(1 - \frac{s_{1a}}{K^2} \right) - 2f_2 \right] \frac{\sqrt{2}}{s_{1b}} k_b \cdot \varepsilon_1 \varepsilon_a \cdot \varepsilon_{\hat{a}} \varepsilon_b \cdot \varepsilon_{\hat{b}} \right\};\end{aligned}\tag{6.7}$$

using

$$f_1(z) \stackrel{z \rightarrow 1}{\sim} \frac{2}{\epsilon^2} c_\Gamma [-\Gamma(1-\epsilon)\Gamma(1+\epsilon)(1-z)^\epsilon - 1 + (1-z)^\epsilon \Gamma(1+\epsilon)\Gamma(1-\epsilon)] = 2f_2,\tag{6.8}$$

and

$$zf_1(z) \stackrel{z \rightarrow 0}{\sim} -\frac{2}{\epsilon^2} c_\Gamma \Gamma(1+\epsilon)\Gamma(1-\epsilon)z^{-\epsilon},\tag{6.9}$$

we obtain ($K^2 = s_{ab}$ in the limit)

$$\begin{aligned}\frac{c_\Gamma}{\epsilon^2} \left(\frac{\mu^2(-s_{ab})}{(-s_{a1})(-s_{1b})} \right)^\epsilon \Gamma(1+\epsilon)\Gamma(1-\epsilon) \varepsilon_a \cdot \varepsilon_{\hat{a}} \varepsilon_b \cdot \varepsilon_{\hat{b}} \left(\frac{\sqrt{2}}{s_{a1}} k_a \cdot \varepsilon_1 - \frac{\sqrt{2}}{s_{1b}} k_b \cdot \varepsilon_1 \right) \\ = -\frac{c_\Gamma}{\epsilon^2} \left(\frac{\mu^2(-s_{ab})}{(-s_{a1})(-s_{1b})} \right)^\epsilon \Gamma(1+\epsilon)\Gamma(1-\epsilon) \varepsilon_a \cdot \varepsilon_{\hat{a}} \varepsilon_b \cdot \varepsilon_{\hat{b}} \text{Soft}^{\text{tree}}(a, 1, b). \\ = -\frac{c_\Gamma}{\epsilon^2} \left(\frac{\mu^2(-s_{ab})}{(-s_{a1})(-s_{1b})} \right)^\epsilon \frac{\pi\epsilon}{\sin \pi\epsilon} \varepsilon_a \cdot \varepsilon_{\hat{a}} \varepsilon_b \cdot \varepsilon_{\hat{b}} \text{Soft}^{\text{tree}}(a, 1, b).\end{aligned}\tag{6.10}$$

The latter form agrees with the result given in ref. [8].

The function we have defined is thus the appropriate generalization of the antenna function to one loop,

$$\text{Ant}^{1\text{-loop}}(\hat{a}, \hat{b} \leftarrow a, 1, b) = L(\hat{a}, \hat{b} \leftarrow a, 1, b).\tag{6.11}$$

With it in hand, we may summarize the singular behavior of a leading-color one-loop color-ordered amplitude whenever the trio of momenta $(a, 1, b)$ become degenerate without either a or b becoming soft as follows,

$$\begin{aligned}A_n^{1\text{-loop}}(1, \dots, a^{\lambda_a}, b^{\lambda_b}, \dots, n) \xrightarrow{\Delta(a, 1, b)/s_{ab}^3 \rightarrow 0} \\ \sum_{\text{ph. pol. } (\hat{a}, \hat{b})} \left(\text{Ant}^{\text{tree}}(\hat{a}, \hat{b} \leftarrow a, 1, b) A_{n-1}^{1\text{-loop}}(1, \dots, -\hat{a}(a, 1, b), -\hat{b}(b, 1, a), \dots, n) \right. \\ \left. + \text{Ant}^{1\text{-loop}}(\hat{a}, \hat{b} \leftarrow a, 1, b) A_{n-1}^{\text{tree}}(1, \dots, -\hat{a}(a, 1, b), -\hat{b}(b, 1, a), \dots, n) \right).\end{aligned}\tag{6.12}$$

As is the case at tree-level, this function contains terms which are not singular in any of the soft or collinear limits. Such terms lead to finite contributions when integrated over ‘singular’ regions of phase space. The inclusion or exclusion of such terms is arbitrary; in a physical observable they will cancel between ‘virtual+singular’ contributions and ‘real-emission’ contributions.

7. Conclusions

The cut-based method has been used extensively in recent years for computations of various gauge-theory amplitudes. We have shown here that it can also be used for simple, direct computations of the universal functions governing the collinear behavior of loop amplitudes, to all orders in the dimensional regulator ϵ . We have also shown how to combine the different singular limits of real emission in one-loop amplitudes — soft and collinear — into a single *antenna* function. A related function introduced by Catani and Seymour [26] for tree-level amplitudes, the dipole factorizing function, has already proven useful in a formalism for handling and cancelling infrared divergences in next-to-leading order calculations. The antenna function presented here should be similarly useful in next-to-next-to-leading order computations of jet processes in perturbative QCD.

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Appendix I. Integrals

In addition to the linear tensor integral, J_3^μ , given in section 3, for the calculations in section 4, we also need the following integrals,

$$\begin{aligned}
J_4^\mu(s_{ab}, z) &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu}{p^2(p-k_a-k_b)^2 p \cdot q} \\
&= \frac{1}{(-s_{ab})^\epsilon q \cdot (a+b)} \left[-\frac{\epsilon}{(1-2\epsilon)} f_2(k_a+k_b)^\mu + \frac{1}{(1-2\epsilon)} \frac{s_{ab}}{2q \cdot (a+b)} f_2 q^\mu \right], \\
J_5^{\mu\nu}(s_{ab}, z) &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu p^\nu}{p^2(p-k_a)^2(p-k_a-k_b)^2 p \cdot q} \\
&= \frac{1}{(-s_{ab})^{1+\epsilon} q \cdot (a+b)} \left[f_{5aa}(z) k_a^\mu k_a^\nu + \frac{1}{2} f_{5ab}(z) (k_a^\mu k_b^\nu + k_b^\mu k_a^\nu) + f_{5bb}(z) k_b^\mu k_b^\nu \right. \\
&\quad + \frac{s_{ab}}{4q \cdot (a+b)} f_{5aq}(z) (k_a^\mu q^\nu + q^\mu k_a^\nu) + \frac{s_{ab}}{4q \cdot (a+b)} f_{5bq}(z) (k_b^\mu q^\nu + q^\mu k_b^\nu) \\
&\quad \left. + \frac{s_{ab}^2}{4(q \cdot (a+b))^2} f_{5qq}(z) q^\mu q^\nu + s_{ab} f_{5g}(z) g^{\mu\nu} \right] \tag{I.1}
\end{aligned}$$

where

$$\begin{aligned}
f_{5aa}(z) &= \frac{(1-\epsilon)}{2(1-2\epsilon)} f_1(z), \\
f_{5ab}(z) &= -\frac{(1-\epsilon)z}{(1-2\epsilon)(1-z)} f_1(z) + \frac{2(1-\epsilon)}{(1-2\epsilon)(1-z)} f_2(z), \\
f_{5bb}(z) &= \frac{(1-\epsilon)z^2}{2(1-2\epsilon)(1-z)^2} f_1(z) - \frac{(\epsilon+z-2\epsilon z)}{(1-2\epsilon)(1-z)^2} f_2(z), \\
f_{5aq}(z) &= -\frac{\epsilon}{(1-2\epsilon)(1-z)} f_1(z) + \frac{2\epsilon}{(1-2\epsilon)(1-z)} f_2(z), \\
f_{5bq}(z) &= -\frac{(1-\epsilon)z}{(1-2\epsilon)(1-z)^2} f_1(z) + \frac{2(1-\epsilon z)}{(1-2\epsilon)(1-z)^2} f_2(z), \\
f_{5qq}(z) &= \frac{(1-\epsilon)}{2(1-2\epsilon)(1-z)^2} f_1(z) - \frac{(2-\epsilon-z)}{(1-2\epsilon)(1-z)^2} f_2(z), \\
f_{5g}(z) &= \frac{z}{4(1-2\epsilon)(1-z)} f_1(z) - \frac{1}{2(1-2\epsilon)(1-z)} f_2(z).
\end{aligned} \tag{I.2}$$

For completeness, we also list the ordinary tensor bubble and triangle integrals of which we make use. These are of course independent of the collinear momentum fraction z , and may therefore be expressed using

f_2 ,

$$\begin{aligned}
I_2 &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-k_a-k_b)^2} \\
&= -\frac{\epsilon}{(1-2\epsilon)} \frac{1}{(-s_{ab})^\epsilon} f_2, \\
I_{2a}^\mu &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu}{p^2(p-k_a-k_b)^2} \\
&= -\frac{\epsilon}{2(1-2\epsilon)} \frac{1}{(-s_{ab})^\epsilon} f_2 (k_a+k_b)^\mu, \\
I_{2b}^{\mu\nu} &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu p^\nu}{p^2(p-k_a-k_b)^2} \\
&= \frac{\epsilon}{2(3-2\epsilon)(1-2\epsilon)} \frac{1}{(-s_{ab})^\epsilon} f_2 \left[\frac{1}{2} s_{ab} g^{\mu\nu} - (2-\epsilon)(k_a+k_b)^\mu (k_a+k_b)^\nu \right], \\
I_3 &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2(p-k_a)^2(p-k_a-k_b)^2} \\
&= \frac{1}{(-s_{ab})^{1+\epsilon}} f_2, \\
I_{3a}^\mu &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu}{p^2(p-k_a)^2(p-k_a-k_b)^2} \\
&= \frac{(1-\epsilon)}{(1-2\epsilon)} (-s_{ab})^{-1-\epsilon} f_2 k_a^\mu - \frac{\epsilon}{(1-2\epsilon)} (-s_{ab})^{-1-\epsilon} f_2 k_b^\mu, \\
I_{3b}^{\mu\nu} &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu p^\nu}{p^2(p-k_a)^2(p-k_a-k_b)^2} \\
&= \frac{1}{(-s_{ab})^{1+\epsilon}} \frac{\epsilon}{4(1-\epsilon)(1-2\epsilon)} s_{ab} f_2 g^{\mu\nu} \\
&\quad + \frac{(2-\epsilon)}{2(1-2\epsilon)} \frac{1}{(-s_{ab})^{1+\epsilon}} f_2 \left(k_a^\mu k_a^\nu - \frac{\epsilon}{1-\epsilon} (k_a^\mu k_b^\nu + k_b^\mu k_a^\nu) - \frac{\epsilon}{2-\epsilon} k_b^\mu k_b^\nu \right), \\
I_{3c}^{\mu\nu\rho} &= -i \int \frac{d^{4-2\epsilon} p}{(2\pi)^{4-2\epsilon}} \frac{p^\mu p^\nu p^\rho}{p^2(p-k_a)^2(p-k_a-k_b)^2} \\
&= \frac{1}{(-s_{ab})^{1+\epsilon}} \frac{\epsilon(2-\epsilon)}{4(1-\epsilon)(1-2\epsilon)(3-2\epsilon)} s_{ab} f_2 (g^{\mu\nu} k_a^\rho + g^{\mu\rho} k_a^\nu + g^{\nu\rho} k_a^\mu) \\
&\quad + \frac{1}{(-s_{ab})^{1+\epsilon}} \frac{\epsilon}{4(1-2\epsilon)(3-2\epsilon)} s_{ab} f_2 (g^{\mu\nu} k_b^\rho + g^{\mu\rho} k_b^\nu + g^{\nu\rho} k_b^\mu) \\
&\quad + \frac{(2-\epsilon)(3-\epsilon)}{2(1-2\epsilon)(3-2\epsilon)} \frac{1}{(-s_{ab})^{1+\epsilon}} f_2 \\
&\quad \times \left(k_a^\mu k_a^\nu k_a^\rho - \frac{\epsilon}{1-\epsilon} (k_a^\mu k_a^\nu k_b^\rho + k_a^\mu k_b^\nu k_a^\rho + k_b^\mu k_a^\nu k_a^\rho) \right. \\
&\quad \left. - \frac{\epsilon}{2-\epsilon} (k_a^\mu k_b^\nu k_b^\rho + k_b^\mu k_a^\nu k_b^\rho + k_b^\mu k_b^\nu k_a^\rho) - \frac{\epsilon}{3-\epsilon} k_b^\mu k_b^\nu k_b^\rho \right).
\end{aligned} \tag{I.3}$$

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